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Quantization of Underdamped, Critically Damped, and Overdamped Electric Circuits With a Power Source

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We have investigated the quantum mechanical effect of the underdamped, critically damped, and overdamped electric circuits with a power source. The charge of the underdamped circuit oscillates while those of the critically damped and overdamped ones don't. The wave function of the system of overdamped circuit represented parabolic cylinder function while underdamped circuit was represented by well-known Hermite polynomial. The eigenvalues of underdamped circuit is discrete while those of the critically damped and overdamped ones are given as continuously.

KEY WORDS: RLC linear circuit; quantization; invariant operator; unitary transformation.

1. INTRODUCTION

The investigation of the harmonic oscillator has been an important task since the early history of physics (Moshinsky and Smirnov (1996), because the most general aspect of physics is vibration that we meet in everyday life. The most standard example of a dissipative system may be a damped harmonic oscillator which is described by a Hamiltonian that is explicitly time-dependent. The quantum mechanical system of damped harmonic oscillator has been of interest in the literature since Kanai (1948) discussed it classically. Since the introduction of a dynamical invariant operator by Lewis (1967) in 1967, the systematic investigation of quantum mechanical time-dependent harmonic oscillator has been facilitated. The main idea to solve the quantum mechanical solution of the time-dependent system is that the wave function is the same as the eigenstate of the dynamical invariant operator, except for some time-dependent phase factor (Lewis and Riesenfeld, 1969). The quantum state of damped harmonic oscillator has been studied with (Oh *et al.*, 1989; Um *et al.*, 1986a,b, 1987, 1996, 1997) and without (Colegrave and

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Abdalla, 1981; Gisin, 1981; Um *et al.*, 1997, 2001a,b; Yeon and Um, 1992) driving force.

The oscillation of charge in *RLC* linear circuit may be a good example of the damped harmonic oscillator. The miniaturization of integrated circuits and components towards atomic scale dimensions demanded the development of quantum theory on a mesoscopic circuit, since the charge carriers such as electrons exhibit quantum mechanical properties while the application of classical mechanics is invalid (Buot, 1993). Nowadays, because of the development of advanced lithography techniques and crystal growth which enable elaborate experiments, the mesoscopic physics and nanoelectronics have rapidly progressed. The generation of squeezing effects for a time-dependent *LC* circuit with a power source has been investigated (Baseia and De Brito, 1993). The effect of circuit parameters on ferroresonant *RLC* circuit are studied by Lamba *et al.* (1998). With the jump on this trend, a quantization for an *RLC* linear circuit with a power source has been tried in the literatures (Chen *et al.*, 1995; Louisell, 1973; Zhang *et al.*, 1998). They obtained the quantum fluctuations of charge and current in the vacuum state and investigated the fluctuations of the charge, the magnetic flux, and the energy of the circuit. Taking account of this direction, we are motivated to study the quantum-mechanical effects of the underdamped, critically damped, and overdamped electric circuits with a power source.

2. HAMILTONIAN AND INVARIANT OPERATOR

By applying Kirchhoff's law, we can obtain the classical equation of motion for charges in *RLC* linear circuit with a power source as

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{\mathcal{E}(t)}{L}, \tag{1}$$

where q is charge, R resistance of the circuit, L inductance, C capacitance, and $\mathcal{E}(t)$ time-dependent power source. We can let the charge $q(t)$ as the analog of the coordinate and the current $p(t)$ as the analog of the momentum. Using Hamilton's equation of motion, we can derive the corresponding Hamiltonian:

$$\hat{H} = e^{-(R/L)t}\frac{\hat{p}^2}{2L} + e^{(R/L)t}\frac{1}{2}\left[\frac{1}{C}\hat{q}^2 - 2\mathcal{E}(t)\hat{q}\right]. \tag{2}$$

To quantize the circuit, we may further pursue the analog between *RLC* linear circuit and mechanical oscillator. The charge $\hat{q}$ and the current $\hat{p}$ are hermitian operators that satisfy the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar. \tag{3}$$

To investigate the quantum mechanical solution of the problem, it is very convenient to introduce an invariant operator. By virtue of its definition, the invariant operator $\hat{I}$ must satisfy the following relation:

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}] = 0. \tag{4}$$

With the substitution of Eq. (2) into Eq. (4), we obtain that

$$\hat{I} = \frac{1}{2}\left\{ \frac{1}{L} e^{-(R/L)t} \left[\hat{p} - p_p(t) + \frac{1}{2} R\, e^{(R/L)t} (\hat{q} - q_p(t)) \right]^2 + L\Omega^2 e^{(R/L)t} (\hat{q} - q_p(t))^2 \right\}, \tag{5}$$

where Ω is given by

$$\Omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2} \right)^{1/2}. \tag{6}$$

and q_p and p_p are the particular solutions of the classical equation of motion in coordinate and momentum space, respectively. Since Eq. (5) is somewhat complicated, let us transform it to more simple form with unitary operator $\hat{U}$ as

$$\hat{I}' = \hat{U}\hat{I}\hat{U}^{\dagger}. \tag{7}$$

In the above equation, we choose $\hat{U}$ as

$$\hat{U} = \hat{U}_3\hat{U}_2\hat{U}_1, \tag{8}$$

where

$$\hat{U}_1 = \exp(iq_p(t)\hat{p}/\hbar)\,\exp(-ip_p(t)\hat{q}/\hbar), \tag{9}$$

$$\hat{U}_2 = \exp[iR\, e^{(R/L)t}\hat{q}^2/(4\hbar)], \tag{10}$$

$$\hat{U}_3 = \exp[-i(R/L)t(\hat{q}\hat{p} + \hat{p}\hat{q})/(4\hbar)]. \tag{11}$$

Then, Eq. (5) can be transformed to

$$\hat{I}' = \frac{\hat{p}^2}{2L} + \frac{1}{2}L\Omega^2\hat{q}^2. \tag{12}$$

The Eq. (12) is not only simply compared to untransformed one but also time-independent so that we can easily deal with it.

3. UNDERDAMPED CIRCUIT

Let us consider an underdamped circuit with $1/C > R^2/(4L)$. The eigenvalue equation for $\hat{I}'$ can be written as

$$\hat{I}'|\phi_n'(t)\rangle = \lambda_n|\phi_n'(t)\rangle. \tag{13}$$

By operating with $\langle\hat{q}|$ from left to both sides of Eq. (13), we can obtain the $\hat{q}$-space eigenstate as

$$\langle\hat{q}\mid\phi_n'(t)\rangle=\sqrt[4]{\frac{L\Omega}{\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{L\Omega}{\hbar}}\hat{q}\right)\exp\left(-\frac{L\Omega}{2\hbar}\hat{q}^2\right), \tag{14}$$

where H_n is nth-order Hermite polynomial. The eigenstate $\langle\hat{q}\mid\phi_n(t)\rangle$ of the untransformed invariant operator is related to the transformed one by the following equation:

$$\langle\hat{q}\mid\phi_n(t)\rangle=\hat{U}^\dagger\langle\hat{q}\mid\phi_n'(t)\rangle. \tag{15}$$

Using Eqs. (8) and (14), Eq. (15) becomes

$$\langle\hat{q}\mid\phi_n(t)\rangle=\sqrt[4]{\frac{L\Omega}{\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left[\sqrt{\frac{L\Omega}{\hbar}}e^{R/(2L)t}(\hat{q}-q_p(t))\right]e^{ip_p(t)\hat{q}/\hbar}$$
$$\times\exp\left[\frac{R}{4L}t-\frac{L}{2\hbar}\left(\Omega+\frac{iR}{2L}\right)e^{(RL)t}(\hat{q}-q_p(t))^2\right]. \tag{16}$$

The wave functions are different from the eigenstate of invariant operator by some time-dependent phase factor, $\exp[i\gamma_n(t)]$ (Lewis and Riesenfeld, 1969).

$$\langle\hat{q}\mid\psi_n(t)\rangle=\langle\hat{q}\mid\phi_n(t)\rangle\exp[i\gamma_n(t)]. \tag{17}$$

By substituting Eq. (17) into Schrödinger equation, we obtain the relation that

$$\hbar\dot{\gamma}_n(t)=\left\langle\phi_n(t)\middle|\left(i\hbar\frac{\partial}{\partial t}-\hat{H}\right)\middle|\phi_n(t)\right\rangle. \tag{18}$$

The right hand side of the above equation can be represented as

$$\left\langle\phi_n'(t)\middle|\hat{U}\left(i\hbar\frac{\partial}{\partial t}-\hat{H}\right)\hat{U}^\dagger\middle|\phi_n'(t)\right\rangle$$
$$=\left\langle\phi_n'(t)\middle|\left[i\hbar\frac{\partial}{\partial t}-\left(\frac{\hat{p}^2}{2L}+\frac{1}{2}L\Omega^2\hat{q}^2\right)\right]\middle|\phi_n'(t)\right\rangle-H_p(q_p(t),p_p(t),t)$$
$$=-\hbar\Omega\left(n+\frac{1}{2}\right)-H_p(q_p(t),p_p(t),t), \tag{19}$$

where

$$H_p(q_p(t),P_p(t),t)=e^{-(R/L)t}\frac{p_p^2}{2L}+e^{(R/L)t}\frac{1}{2}\left[\frac{1}{C}q_p^2-2\mathcal{E}(t)q_p\right]. \tag{20}$$

By inserting Eq. (19) into Eq. (18), we can obtain the γ_n as

$$\gamma_n=-\Omega t\left(n+\frac{1}{2}\right)-\frac{1}{\hbar}\int_0^t H_p\big(q_p(t'),p_p(t'),t'\big)dt'. \tag{21}$$

Using Eqs. (16) and (21), the Eq. (17) can be rewritten as

$$\langle \hat{q} \mid \psi_n(t)\rangle = \sqrt[4]{\frac{L\Omega}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} H_n\left[\sqrt{\frac{L\Omega}{\hbar}} e^{R/(2L)t}(\hat{q} - q_p(t))\right] e^{i p_p(t)\hat{q}/\hbar}$$
$$\times \exp\left[\frac{Rt}{4L} - \frac{L}{2\hbar}\left(\Omega + \frac{iR}{2L}\right) e^{(R/L)t}(\hat{q} - q_p(t))^2\right]$$
$$\times \exp\left[-i\Omega t\left(n + \frac{1}{2}\right) - \frac{i}{\hbar}\int_0^t H_p\big(q_p(t'), p_p(t'), t'\big)dt'\right]. \tag{22}$$

This is the full wave function that satisfies the Schrödinger equation. The eigenvalue is given discretely because the system is oscillatory and bounded.

The classical energy of the system is given by

$$E = \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2. \tag{23}$$

And, the quantum mechanical energy can be defined as

$$E_n = e^{-(2R/L)t}\frac{1}{2L}\big\langle\psi_n(t)\big|\hat{p}^2\big|\psi_n(t)\big\rangle + \frac{1}{2C}\big\langle\psi_n(t)\big|\hat{q}^2\big|\psi_n(t)\big\rangle. \tag{24}$$

Using Eq. (22), Eq. (24) can be calculated as

$$E_n = e^{-(R/L)t}\hbar\frac{1}{\Omega LC}\left(n + \frac{1}{2}\right) + E_p(q_p(t), \dot{q}_p(t)), \tag{25}$$

where

$$E_p(q_p(t), \dot{q}_p(t)) = \frac{1}{2}L\dot{q}_p^2(t) + \frac{1}{2C}q_p^2(t), \tag{26}$$

$$\dot{q}_p(t) = \frac{p_p(t)}{L}e^{-(R/L)t}. \tag{27}$$

In Eq. (25), the first term disappears as time goes by while the second term remains.

4. CRITICALLY DAMPED CIRCUIT

Now, we consider critically damped circuit that the condition is given by $1/C = R^2/(4L)$. Then, Eq. (12) is simplified to

$$\hat{I}' = \frac{\hat{p}^2}{2L}. \tag{28}$$

The Eq. (28) is same with the Hamiltonian of free particle. Let us write the eigenvalue equation of Eq. (28) as

$$\hat{I}'|\phi'(t)\rangle = \lambda|\phi'(t)\rangle. \tag{29}$$

Then, the $\hat{q}$-space eigenstate can be calculated as

$$\langle \hat{q} \mid \phi'(t)\rangle = C_1 \exp\left(\frac{i}{\hbar}\sqrt{2L\lambda}\,\hat{q}\right) + C_2 \exp\left(-\frac{i}{\hbar}\sqrt{2L\lambda}\,\hat{q}\right), \tag{30}$$

where C_1 and C_2 are integral constants. The eigenstate of the untransformed invariant operator is given by

$$\begin{aligned}\langle \hat{q} \mid \phi(t)\rangle &= \hat{U}^\dagger \langle \hat{q} \mid \phi'(t)\rangle \\ &= \exp\left[-\frac{iR}{4\hbar}\, e^{(R/L)t}(\hat{q} - q_p(t))^2\right] e^{ip_p(t)\hat{q}/\hbar}\, e^{R/(4L)t} \\ &\quad \times \left[C_1 \exp\left(\frac{i}{\hbar}\sqrt{2L\lambda}\, e^{R/(2L)t}(\hat{q} - q_p(t))\right)\right. \\ &\quad \left.\times\, C_2 \exp\left(-\frac{i}{\hbar}\sqrt{2L\lambda}\, e^{R/(2L)t}(\hat{q} - q_p(t))\right)\right]. \end{aligned} \tag{31}$$

We can denote the phase factor of the wave function as $\exp\,[i\gamma(t)]$:

$$\langle \hat{q} \mid \psi(t)\rangle = \langle \hat{q} \mid \phi(t)\rangle \exp[i\gamma(t)]. \tag{32}$$

By substitution of Eq. (32) into Schrödinger equation, we obtain that

$$\hbar\dot{\gamma}(t) = \left\langle \phi(t) \middle| \left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right) \middle| \phi(t) \right\rangle. \tag{33}$$

The right hand side of Eq. (33) can be calculated as

$$\begin{aligned}&\left\langle \phi'(t) \middle| \hat{U}\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)\hat{U}^\dagger \middle| \phi'(t) \right\rangle \\ &= \left\langle \phi'(t) \middle| \left(i\hbar\frac{\partial}{\partial t} - \frac{\hat{p}^2}{2L}\right) \middle| \phi'(t) \right\rangle - H_p(q_p(t), p_p(t), t) \\ &= -\lambda - H_p(q_p(t), p_p(t), t). \end{aligned} \tag{34}$$

From Eqs. (33) and (34), we can obtain $\gamma(t)$ as

$$\gamma(t) = -\frac{\lambda}{\hbar}t - \frac{1}{\hbar}\int_0^t H_p\big(q_p(t)', p_p(t'), t'\big)dt'. \tag{35}$$

Substitution of Eqs. (31) and (35) into Eq. (32), we can right the full wave function as

$$\begin{aligned}\langle \hat{q} \mid \psi(t)\rangle &= \exp\left[-\frac{iR}{4\hbar}\, e^{(R/L)t}(\hat{q} - q_p(t))^2\right] e^{ip_p(t)\hat{q}/\hbar}\, e^{R/(4L)t} \\ &\quad \times \left[C_1 \exp\left(\frac{i}{\hbar}\sqrt{2L\lambda}\; e^{R/(2L)t}(\hat{q} - q_p(t))\right)\right.\end{aligned}$$

$$+C_2 \exp\left(-\frac{i}{\hbar}\sqrt{2L\lambda}\; e^{R/(2L)t}(\hat{q} - q_p(t))\right)\Bigg]$$

$$\times \exp\left[-i\frac{\lambda}{\hbar}t - \frac{i}{\hbar}\int_0^t H_p(q_p(t'), p_p(t'), t')\,dt'\right]. \qquad (36)$$

Note that the wave function, Eq. (36), is same with that of unbounded one and the eigenvalue is given continuously.

5. OVERDAMPED CIRCUIT

Let us now consider for critically damped circuit that the condition is given by $1/C < R^2/(4L)$. Then, Eq. (12) can be rewritten as

$$\hat{I}' = \frac{\hat{p}^2}{2L} - \frac{1}{2}L\tilde{\Omega}^2\hat{q}^2, \qquad (37)$$

where

$$\tilde{\Omega}^2 \equiv \frac{R^2}{4L^2} - \frac{1}{LC} = -\Omega^2 > 0. \qquad (38)$$

Note that, in this case, $\tilde{\Omega}^2$ is always positive. The Eq. (37) is same with the Hamiltonian of the harmonic parabola potential system. The eigenvalue equation for $\hat{I}'$ may be written as

$$\hat{I}'|\phi'(t)\rangle = \lambda|\phi'(t)\rangle. \qquad (39)$$

Substitution of Eq. (37) into Eq. (39) and operating $\langle\hat{q}|$ from left to both sides of the equation, we obtain that

$$\frac{\partial^2\langle\hat{q} \mid \phi'(t)\rangle}{\partial\hat{Q}^2} + \left(\prod + \frac{1}{4}\hat{Q}^2\right)\langle\hat{q} \mid \phi'(t)\rangle = 0, \qquad (40)$$

where

$$\hat{Q} = \sqrt{\frac{2L\tilde{\Omega}}{\hbar}}\hat{q}, \qquad (41)$$

$$\prod = \frac{\lambda}{\hbar\tilde{\Omega}}. \qquad (42)$$

The solution of Eq. (40) is given by

$$\langle\hat{q} \mid \phi'(t)\rangle = C_1' D_{-i\prod - 1/2}\left(\frac{1+i}{\sqrt{2}}\hat{Q}\right) + C_2' D_{-i\prod - 1/2}\left(-\frac{1+i}{\sqrt{2}}\hat{Q}\right), \qquad (43)$$

where $D_\nu(z)$ is parabolic cylinder function which is defined as (Erdély, 1953a)

$$D_\nu(z) = 2^{(\nu-1)/2}\,\exp(-z^2/4)z\Psi(1/2 - \nu/2, 3/2; z^2/2),$$

$$\Psi(a, c; y) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-yt}\, t^{a-1}(1+t)^{c-a-1}\,dt. \qquad (44)$$

In fact, $D_\nu(z)$ is a solution of the following differential equation (Erdély, 1953b)

$$\frac{d^2\phi}{dz^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right)\phi = 0. \tag{45}$$

The eigenstate of the untransformed invariant operator is given by

$$\begin{aligned}\langle \hat{q} \mid \phi(t)\rangle &= \hat{U}^\dagger \langle \hat{q} \mid \phi'(t)\rangle \\ &= \exp\left[-\frac{iR}{4\hbar}\, e^{(R/L)t}(\hat{q} - q_p(t))^2\right] e^{ip_p(t)\hat{q}/\hbar}\, e^{R/(4L)t} \\ &\quad \times \left[C_1' D_{-i\prod -1/2}\left(\frac{1+i}{\sqrt{2}}\, e^{R/(2L)t}\,\hat{Q}'\right)\right. \\ &\quad \left. + C_2' D_{-i\prod -1/2}\left(-\frac{1+i}{\sqrt{2}}\, e^{R/(2L)t}\,\hat{Q}'\right)\right],\end{aligned} \tag{46}$$

where

$$\hat{Q}' = \sqrt{\frac{2L\tilde{\Omega}}{\hbar}}(q - q_p(t)). \tag{47}$$

The wave function again can be written as

$$\langle \hat{q} \mid \psi(t)\rangle = \langle \hat{q} \mid \phi(t)\rangle\ \exp[i\gamma(t)]. \tag{48}$$

By the same process with the previous case, we obtain $\gamma(t)$ as

$$\gamma(t) = -\frac{\lambda}{\hbar}t - \frac{1}{\hbar}\int_0^t H_p(q_p(t'),\, p_p(t'),\, t')\, dt'. \tag{49}$$

By substituting Eqs. (46) and (49) into Eq. (48), we can obtain that

$$\begin{aligned}\langle \hat{q} \mid \psi(t)\rangle &= \exp\left[-\frac{iR}{4\hbar}\, e^{(R/L)t}(\hat{q} - q_p(t))^2\right] e^{ip_p(t)\hat{q}/\hbar}\, e^{R/(4L)t}\left[C_1' D_{-i\prod -1/2}\right. \\ &\quad \left.\left(\frac{1+i}{\sqrt{2}}\, e^{R/(2L)t}\,\hat{Q}'\right) + C_2' D_{-i\prod -1/2}\left(-\frac{1+i}{\sqrt{2}}\, e^{R/(2L)t}\,\hat{Q}'\right)\right] \\ &\quad \times\ \exp\left[-\frac{i\lambda}{\hbar}t - \frac{i}{\hbar}\int_0^t H_p(q_p(t'),\, p_p(t'),\, t')\, dt'\right].\end{aligned} \tag{50}$$

The wave function Eq. (50) is also continuous as that of critically damped one.

6. SUMMARY

Using the advantage of invariant operator, we derived the quantum mechanical solution of the *RLC* linear circuit. We made use of the advantage of invariant operator to derive the solution of Schrödinger equation for underdamped, critically damped, and overdamped circuits. The transformed invariant operator expressed with modified frequency Ω, and same with the Hamiltonian of standard harmonic oscillator for underdamped circuit, of free particle for critically damped one, and of the harmonic parabola potential system for overdamped one. The wave function of the system of underdamped circuit was represented by well-known Hermite polynomial. On the other hand, the wave function of overdamped circuit represented parabolic cylinder function. The charge of the underdamped circuit oscillates while those of the critically damped and overdamped ones don't. The eigenvalues of underdamped circuit is discrete while those of the critically damped and overdamped ones are given as continuously.

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